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ON INVARIANT INTEGRALS IN THE MARGUERRE-VON KÁRMÁN SHALLOW SHELL

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Abstract—By employing the second-order Noether's theorem, several new invariant integrals have been derived for the non-linear shallow shell—the Marguerre-von Kármán shell. The dynamic effect is considered in the derivations. These invariant integrals are path-independent over the projection image of the middle surface of the shell in a Cartesian plane, in which the projection area of the middle surface of the shallow shell is maximum. The proposed invariant integrals can be used to evaluate the asymptotic field around a defect embedded in the shell. Unlike most other studies, the Lagrangian density of the invariant variational principle used here belongs to a mixed type variational principle. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Since Eshelby first introduced his energy-momentum tensor in 1956, the conservation laws in elasticity theory have been thoroughly studied in the last three decades by, for example, Eshelby (1956, 1975), Rice (1968), Knowles and Sternberg (1972), Fletcher (1976), Edelen (1981), Delph (1982), Olver (1984a, 1984b, 1988), Eischen and Herrmann (1987), Suhubi (1989), Yeh *et al.* (1993a, 1993b) and others.

A main reason for this seemingly ever-lasting interest in the subject is that some of these path-independent integrals can be related to the material properties near the vicinity of a defect embedded in the continuum. The "J-integral" proposed by Rice (1968) exemplified how powerful these invariant integrals can be in the application of fracture mechanics.

Knowles and Sternberg (1972) and Fletcher (1976) systematically studied the conservation laws of elasticity by using the first-order Noether's theorem, which mainly consider the invariant properties of variational principles under a group of infinitesimal transformations. Although later Olver (1984a, 1984b, 1988) used Lie group theory to categorize all the invariance transformations and invariant integrals in linear elasticity, in practice, the **J**, **L**, and M integrals remain as the most important invariant integrals in applications. This is largely due to the fact that they represent some basic, intrinsic symmetry properties of an isotropic continuous medium.

However, how to apply the conservation laws of three dimensional (3-D) elasticity to engineering problems seems to be a nontrivial task, because, in reality, most practical engineering problems are modeled by specific structural theories, such as the theory of plates and shells, instead of 3-D elasticity itself.

There are two different ways to approach the problem. The first approach takes invariant integrals of 3-D elasticity for granted and inserts particular forms of energy, or energy rate of the structure into those invariant integrals of 3-D elasticity, such as Bergez (1976), Nicholson and Simmonds (1980), Simmonds and Duva (1981). The main hypothesis behind this procedure is that the structure theory is viewed only as a degeneracy of general

3-D elasticity. There may be some truths in doing so, but the effort or the attitude may not be always appreciated by both down-to-earth engineers and rigorous mathematicians. The reason is simple: if an engineer can use 3-D elasticity formulae to analyze a structural component, why does he need the theory of plates and shells at all? On the other hand, if one adopts the theory of plates and shells in the analysis, due to the fact that almost all the structural theories involve *a priori* assumptions, the path-independent integral of 3-D elasticity is most likely not the actual invariant integrals for the particular differential equations that describe the behaviour of the particular type of structures involved. Precisely speaking, the invariant integrals in 3-D elasticity may not be compatible with the commonly used plate and shell theories. Indeed, those approximated or analogous integrals are usually not path-independent in the theories of plates or shells. As pointed out by Sosa and Herrmann (1989), a group of analogous invariant integrals proposed by Bergez (1976) for linear shell theory are not path-independent in the shell theory at all. In fact, most of the established structure theories are a set of closed, self-consistent mathematical system rather than just a degeneracy of 3-D elasticity. Therefore, the degeneracy approach of invariant integral for structural theory has its generic deficiency.

The second approach is a series work conducted by G. Herrmann, H. Sosa and their colleagues (Sosa *et al.*, 1988; Sosa and Herrmann, 1989; Chien *et al.*, 1993); their philosophy relies upon the fact that the partial differential equations that govern the motions of a particular class of structures may yield their own invariant integrals. Therefore, it would be both technically significant and aesthetically appealing to establish the conservation laws based on the original structural theory, as the counterparts to complement those in 3-D elasticity. From this perspective, we believe, Sosa *et al.* (1988) first derived the three basic types of invariant integrals for Reissner–Mindlin plate theory via an efficient procedure, which was proposed early by Eischen and Herrmann (1987).

Up to today, to the authors' knowledge, the issue on invariant integrals of nonlinear plate theory and shell theory in general are still open and far away from being completely resolved. In this paper, we present some new results about invariant integrals for a class of nonlinear shallow shell—the Marguerre-von Kármán theory, hoping to add some contribution to this matter. In the paper, the technical term, nonlinear shallow shell, is also used as the synonym of the initially deflected non-linear plate.

It should be noted that since Olver's work (1986), using the tool of Lie group and Lie algebra has become a standard technique in deriving invariant integrals from a variational principle, owing to the fact that the method possesses remarkable simplicity, and it approaches the matter in a systematic manner. Nevertheless, it may, sometimes, obscure the physical meanings of a mathematical operation. In order to keep a clear picture in physics, we adopt an old fashioned, engineering type derivation in this paper, such that it can be easily accessible, and more useful in engineering applications.

2. PRELIMINARIES

Most studies conducted on conservation laws of elasticity are primarily based on the Noether's celebrated theorem (Noether, 1918), which was demonstrated only for the first-order variational problem, although a higher order, abstract expression was outlined in principle. For the conventional elasticity theory, the Lagrangian density only involved with the first-order derivatives of the displacements, thus, the application of Noether's theorem is straight forward. On the other hand, the curvature—the primary variable in the theory of plates and shells—is expressed in terms of the second order derivatives of the deflection of a plate, or a shell; consequently, the associated variational problem is a second-order variational problem. Thus, the derivation procedure of invariant integrals based on the first-order Noether theorem needs to be modified and extended to encompass this general case. It should be mentioned that the method that was employed by Sosa *et al.* (1988) has been only used to derive invariant integrals for linear plate theories, and since the Marguerre-von Kármán theory is a nonlinear shell theory, there might be some difficulties to apply the method in this particular case, if not entirely impossible.

In fact, Noether's theorem was indeed extended to the second-order variational problem (Logan and Blakeslee, 1975; Blakeslee and Logan, 1976, 1977). From the pure mathematics view point, the extension is straight forward and elementary; however, from the physics, or application standpoint, for example the nonlinear shallow shell theory in this case, it certainly deserves special attention, and merits an independent treatment. For the sake of easy reference, we shall first outline the generalized version of Noether's theorem and the Marguerre-von Kármán shallow shell theory; and then we will derive the conservation laws of the nonlinear shallow shell by applying the second order Noether's theorem.

2.1. *The second-order Noether Theorem*

Let $D \subset \mathbb{R}^n$ be a single, simply-connected region. We consider the following second-order fundamental integral,

$$J(\mathbf{q}) = \int_D L(\mathbf{x}, \mathbf{q}, \partial\mathbf{q}, \partial^2\mathbf{q})\, d\mathbf{x}, \tag{1}$$

where $\mathbf{x} := (x_1, \ldots, x_n)$, $d\mathbf{x} := dx_1\, dx_2 \ldots dx_n$, $\mathbf{q} := (q^1(\mathbf{x}), \ldots, q^m(\mathbf{x}))$, and $\mathbf{q}(\mathbf{x}) \in \mathscr{V}$, $\mathscr{V} \subset C^4_m(D)$. Note that

$$C^4_m(D) := \underbrace{C^4(D) \times C^4(D) \ldots C^4(D)}_{m}. \tag{2}$$

The notation $\partial\mathbf{q}(\mathbf{x})$ and $\partial^2\mathbf{q}(\mathbf{x})$ denote the collection of the first order and the second order derivatives of $\mathbf{q}$, i.e.

$$\partial\mathbf{q} := \left\{ q^k_{,i} := \frac{\partial q^k}{\partial x_i} \middle| 1 \leqslant k \leqslant m, \quad 1 \leqslant i \leqslant n \right\}, \tag{3}$$

$$\partial^2\mathbf{q} := \left\{ q^k_{,ij} := \frac{\partial^2 q^k}{\partial x_i \partial x_j} \middle| 1 \leqslant k \leqslant m, \quad 1 \leqslant i,j \leqslant n \right\}. \tag{4}$$

Let the field undergoing the following r-parameter family transformation,

$$\bar{\mathbf{x}} = \boldsymbol{\phi}(\mathbf{x}, \mathbf{q}, \boldsymbol{\varepsilon}), \quad \bar{\mathbf{q}} = \boldsymbol{\psi}(\mathbf{x}, \mathbf{q}, \boldsymbol{\varepsilon}). \tag{5}$$

where $\boldsymbol{\varepsilon} := (\varepsilon_1, \ldots, \varepsilon_r)$ is a r-parameter family.

Moreover, it is assumed that the transforms (5) are always uniformly continuous around the origin of $\boldsymbol{\varepsilon}$. Specifically,

$$\boldsymbol{\phi}(\mathbf{x}, \mathbf{q}, 0) = \mathbf{x} \quad \text{and} \quad \boldsymbol{\psi}(\mathbf{x}, \mathbf{q}, 0) = \mathbf{q}; \tag{6}$$

or in component form,

$$\phi_i(\mathbf{x}, \mathbf{q}, 0) = x_i \quad \text{and} \quad \psi^k(\mathbf{x}, \mathbf{q}, 0) = q^k. \tag{7}$$

The associated infinitesimal generators of the transformations (5) are defined as follows

$$\tau^i_s(\mathbf{x}, \mathbf{q}) := \left.\frac{\partial \phi_i}{\partial \varepsilon_s}\right|_{\boldsymbol{\varepsilon}=0}, \quad \xi^k_s(\mathbf{x}, \mathbf{q}) := \left.\frac{\partial \psi^k}{\partial \varepsilon_s}\right|_{\boldsymbol{\varepsilon}=0}, \tag{8}$$

where $1 \leqslant i \leqslant n$, $1 \leqslant k \leqslant m$, and $1 \leqslant s \leqslant r$.

Definition 2.1. The fundamental integral (1) is said to be invariant under the r-parameter family of transformations (5), if and only if

$$\int_{\bar{D}} L(\bar{\mathbf{x}}, \bar{\mathbf{q}}, \partial\bar{\mathbf{q}}, \partial^2\bar{\mathbf{q}})\, \mathrm{d}\bar{\mathbf{x}} = \int_D L(\mathbf{x}, \mathbf{q}, \partial\mathbf{q}, \partial^2\mathbf{q})\, \mathrm{d}\mathbf{x}. \tag{9}$$

Moreover, the fundamental integral (1) is said to be infinitesimally invariant, if and only if $\exists\Phi_s^i$ such that

$$\int_{\bar{D}} L(\bar{\mathbf{x}}, \bar{\mathbf{q}}, \partial\bar{\mathbf{q}}, \partial^2\bar{\mathbf{q}})\, \mathrm{d}\bar{\mathbf{x}} - \int_D L(\mathbf{x}, \mathbf{q}, \partial\mathbf{q}, \partial\mathbf{q})\, \mathrm{d}\mathbf{x} = o(\boldsymbol{\varepsilon}) + \varepsilon_s \int_D \frac{\partial}{\partial x_i}\Phi_s^i\, \mathrm{d}\mathbf{x}. \tag{10}$$

Remark 2.1. (1) The definition of infinitesimally invariant integrals given in eqn (10) is taking into account a null class of Lagrangian function, Φ_s^i. There is, however, a danger coming out from this generalization†: whenever the variational problem involves with Neumann boundary condition, the definition (10) is no longer valid, because the associated Neumann boundary data will change due to the presence of the null Lagrangian. This was, probably, first noted by Courant and Hilbert (1953), and it was further elaborated in detail by both Edelen (1981, 1985) and Olver (1983). Nevertheless, by taking into account the null Lagrangian, additional conservation laws may be found under restriction; such as those found in elastostatics by Delph (1982). In this paper, we only consider the case $\Phi_s^i \equiv 0$. (2) Clearly, if the fundamental integral $J(\mathbf{q})$ is invariant, it must be infinitesimally invariant; nonetheless, the converse is not true.

The following theorems are the main technical ingredient of this work, which are the second-order Noether theorem derived by Logan and Blakeslee. Since the proofs are elementary and calculus in nature, they are outlined right after the theorems are stated. For detail information, readers may consult Logan (1977).

Theorem 2.1. (Logan and Blakeslee) If the fundamental integral $J(\mathbf{q})$ (1) is infinitesimally invariant under the r-parameter family transformation (5), the Lagrangian density and its derivatives satisfy the following r identities,

$$\frac{\partial L}{\partial x_i}\tau_s^i + \frac{\partial L}{\partial q^k}\xi_s^k + \frac{\partial L}{\partial q_{,i}^k}\left(\frac{\partial \xi_s^k}{\partial x_i} - q_{,j}^k\frac{\partial \tau_s^j}{\partial x_i}\right) + \frac{\partial L}{\partial q_{,ij}^k}\left(\frac{\partial^2 \xi_s^k}{\partial x_i \partial x_j} - q_{,jl}^k\frac{\partial \tau_s^l}{\partial x_i} - q_{,il}^k\frac{\partial \tau_s^l}{\partial x_j} - q_{,l}^k\frac{\partial^2 \tau_s^l}{\partial x_i \partial x_j}\right) + L\frac{\partial \tau_s^i}{\partial x_i} = \frac{\partial \Phi_s^i}{\partial x_i}. \tag{11}$$

The proof of the theorem is just taking the derivative of ε_s with respect to (10).

Theorem 2.2. (Noether) Let $\mathbf{q} \in C_m^4(D)$ and $\mathbf{q}$ satisfying the following second-order Euler-Lagrangian equation

$$E_k^{(2)} := \frac{\partial L}{\partial q^k} - \frac{\partial}{\partial x_i}\frac{\partial L}{\partial q_{,i}^k} + \frac{\partial^2}{\partial x_i \partial x_j}\frac{\partial L}{\partial q_{,ij}^k} = 0. \tag{12}$$

† This has been pointed out to us by an anonymous referee.

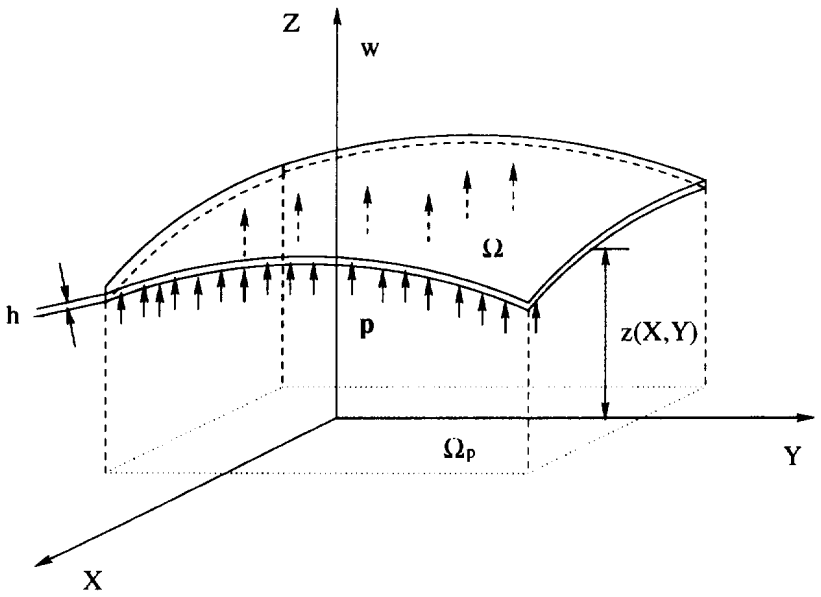

Fig. 1. Configuration of a nonlinear shallow shell.

If the fundamental integral (1) is invariant under the r-parameter family transformations (5), then the following r-conservation laws hold true,

$$\frac{\partial}{\partial x_i}\left[L\tau_s^i+\frac{\partial L}{\partial q_{,i}^k}C_s^k+\frac{\partial L}{\partial q_{,ij}^k}\frac{\partial C_s^k}{\partial x_j}-\frac{\partial}{\partial x_j}\frac{\partial L}{\partial q_{,ij}^k}C_s^k-\Phi_s^i\right]=-E_k^{(2)}C_s^k=0, \tag{13}$$

where $C_s^k := \xi_s^k - q_{,l}^k\tau_s^l$.

Proof. With some rearrangement, one may rewrite (11) as follows

$$\frac{\partial}{\partial x_i}(L\tau_s^i)+\frac{\partial L}{\partial q^k}C_s^k+\frac{\partial L}{\partial q_{,i}^k}\frac{\partial C_s^k}{\partial x_i}+\frac{\partial L}{\partial q_{,ij}^k}\frac{\partial^2 C_s^k}{\partial x_i\partial x_j}=\frac{\partial\Phi_s^i}{\partial x_i}. \tag{14}$$

Considering

$$\frac{\partial L}{\partial q_{,i}^k}\frac{\partial C_s^k}{\partial x_i}=\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial q_{,i}^k}C_s^k\right)-\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial q_{,i}^k}\right)C_s^k \tag{15}$$

$$\frac{\partial L}{\partial q_{,ij}^k}\frac{\partial^2 C_s^k}{\partial x_i\partial x_j}=\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial q_{,ij}^k}\frac{\partial C_s^k}{\partial x_j}-\frac{\partial}{\partial x_j}\frac{\partial L}{\partial q_{,ij}^k}C_s^k\right)+\frac{\partial^2}{\partial x_i\partial x_j}\frac{\partial L}{\partial q_{,ij}^k}C_s^k, \tag{16}$$

one can readily show

$$\begin{aligned}\frac{\partial}{\partial x_i}\left(L\tau_s^i+\frac{\partial L}{\partial q_{,i}^k}C_s^k+\frac{\partial L}{\partial q_{,ij}^k}\frac{\partial C_s^k}{\partial x_j}-\frac{\partial}{\partial x_j}\frac{\partial L}{\partial q_{,ij}^k}C_s^k-\Phi_s^i\right)&=-\left(\frac{\partial L}{\partial q^k}-\frac{\mathrm{d}}{\mathrm{d}x_i}\frac{\partial L}{\partial q_{,i}^k}+\frac{\partial^2}{\partial x_i\partial x_j}\frac{\partial L}{\partial q_{,ij}^k}\right)C_s^k\\&=E_k^{(2)}C_s^k=0.\end{aligned} \tag{17}$$

2.2. *The Marguerre-von Kármán shallow shell theory*

In this section, we shall briefly outline the Marguerre-von Kármán shallow shell theory to supply the ground information. There are quite a few standard documents about the

Marguerre-von Kármán shallow shell theory, and our reference is mainly taken from Marguerre (1938), Chia (1977), and Washizu (1975). To facilitate the later derivation, the notations are slightly changed.

Take the x–y plane of a three-dimensional Cartesian basis as the reference plane. The undeformed middle surface of the shallow shell is described as

$$z = z(x, y). \tag{18}$$

The actual displacements of a point (x, y) in the middle surface are denoted as u, v, w. Subsequently, with some *a priori* assumptions, the corresponding strain components in the middle surface can be deduced as follows

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial w}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \tag{19}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial w}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \tag{20}$$

$$2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}. \tag{21}$$

We denote the following strain components as the "in-plane membrane strain",

$$\varepsilon_{xx}^{(0)} := \frac{\partial u_x^{(0)}}{\partial x} := \frac{\partial u}{\partial x}, \tag{22}$$

$$\varepsilon_{yy}^{(0)} := \frac{\partial u_y^{(0)}}{\partial y} := \frac{\partial v}{\partial y}, \tag{23}$$

$$\varepsilon_{xy}^{(0)} := \frac{1}{2}\left(\frac{\partial u_x^{(0)}}{\partial y} + \frac{\partial u_y^{(0)}}{\partial x}\right) := \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{24}$$

In the Marguerre-von Kármán shallow shell theory, the curvatures of the shell still remain the same as those in the linear plate theories,

$$\kappa_{xx} = \frac{\partial^2 w}{\partial x^2}, \quad \kappa_{yy} = \frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} = \frac{\partial^2 w}{\partial x \partial y}. \tag{25}$$

Introduce the Airy stress function such that the stress resultants can be expressed as

$$N_{xx} = \frac{\partial^2 F}{\partial y^2}, \quad N_{yy} = \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}. \tag{26}$$

Then, one can readily verify the following relationships between the "in-plane membrane strain" and the Airy stress functions along with the gradient of plate's deflection:

$$\varepsilon_{xx}^{(0)} = -\frac{1}{Eh}\left(\nu\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2}\right) - \frac{\partial z}{\partial x}\frac{\partial w}{\partial x} - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \tag{27}$$

$$\varepsilon_{yy}^{(0)} = -\frac{1}{Eh}\left(-\frac{\partial^2 F}{\partial x^2} + \nu\frac{\partial^2 F}{\partial y^2}\right) - \frac{\partial z}{\partial x}\frac{\partial w}{\partial y} - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \tag{28}$$

$$\varepsilon_{xy}^{(0)} = -\frac{(1+\nu)}{Eh}\frac{\partial^2 F}{\partial x \partial y} - \frac{1}{2}\left(\frac{\partial z}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial w}{\partial x}\right) - \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}. \tag{29}$$

The constitutive relations between the stress couples and curvatures are listed as follows

$$M_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \tag{30}$$

$$M_{yy} = -D\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \tag{31}$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}, \tag{32}$$

where $D = Eh^3/12(1-\nu)$.

The equations of motion of the nonlinear shallow shell read as

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \bar{X} = 0, \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + \bar{Y} = 0, \tag{33}$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + \frac{\partial}{\partial x}\left\{N_{xx}\left(\frac{\partial z}{\partial x} + \frac{\partial w}{\partial x}\right) + N_{xy}\left(\frac{\partial z}{\partial y} + \frac{\partial w}{\partial y}\right)\right\}$$
$$+ \frac{\partial}{\partial y}\left\{N_{xy}\left(\frac{\partial z}{\partial x} + \frac{\partial w}{\partial x}\right) + N_{yy}\left(\frac{\partial z}{\partial y} + \frac{\partial w}{\partial y}\right)\right\} + \bar{p} = \rho h\frac{\partial^2 w}{\partial t^2}, \tag{34}$$

where $\bar{X}$ and $\bar{Y}$ are the components of in-plane external force distribution per area. It should be noted that, in this paper, it is always assumed that the normal external load distribution $\bar{p}$ is uniform.

To derive a variational principle concerning the above system with the independent variable F and w, a formal procedure is suggested by Washizu (1975). A mixed, three-field variable variational principle for the Marguerre-von Kármán shallow shell may be proposed as follows

$$\Pi_{III} = \int_{\Omega_p}\left\{\frac{Eh}{2(1-\nu^2)}[(\varepsilon_{xx}+\varepsilon_{yy})^2 + 2(1-\nu)(\varepsilon_{xy}^2 - \varepsilon_{xx}\varepsilon_{yy})]\right.$$
$$+ \frac{D}{2}[(\kappa_{xx}+\kappa_{yy})^2 + 2(1-\nu)(\kappa_{xy}^2 - \kappa_{xx}\kappa_{yy})] - \bar{X}u - \bar{Y}v - \bar{p}w$$
$$- \left[\varepsilon_{xx} - u_{,x} - z_{,x}w_{,x} - \frac{1}{2}w_{,x}^2\right]N_{xx} - \left[\varepsilon_{yy} - v_{,y} - z_{,y}w_{,y} - \frac{1}{2}w_{,y}^2\right]N_{yy}$$
$$- \left[\varepsilon_{xy} - \frac{1}{2}(u_{,y} + v_{,x} + z_{,y}w_{,x} + z_{,x}w_{,y} + w_{,x}w_{,y})\right]N_{xy}$$
$$\left. + (\kappa_{xx} - w_{,ww})M_{xx} + (\kappa_{yy} - w_{,yy})M_{yy} + 2(\kappa_{xy} - w_{,xy})M_{xy}\right\}\mathrm{d}x\,\mathrm{d}y$$
$$+ \oint_{\partial\Omega_p}[\text{The terms on the boundary}]\,\mathrm{d}\Gamma, \tag{35}$$

where the integral area, Ω_p, is the projection of the middle surface Ω onto the x–y plane.

Consider the stationary conditions,

$$N_{xx} = \frac{Eh}{(1-\nu^2)}(\varepsilon_{xx} + \nu\varepsilon_{yy}), \tag{36}$$

$$N_{yy} = \frac{Eh}{(1-\nu^2)}(\nu\varepsilon_{xx} + \varepsilon_{yy}), \tag{37}$$

$$N_{xy} = \frac{Gh}{2}\varepsilon_{xy}, \tag{38}$$

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \bar{X} = 0, \tag{39}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + \bar{Y} = 0, \tag{40}$$

and substitute eqn (25) and (26) into the functional (35). It then yields a new functional,

$$\Pi^* := \int_{\Omega_p} W \,\mathrm{d}\Omega + \oint_{\partial\Omega_p} [\text{The boundary terms}]\,\mathrm{d}\Gamma \tag{41}$$

where

$$\begin{aligned} W := & -\frac{1}{2Eh}\left[\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)^2 + 2(1-\nu)\left(\left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 - \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2}\right)\right] \\ & + \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\nu)\left(\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)\right] \\ & + \left[\frac{\partial^2 F}{\partial y^2}\left(\frac{\partial z}{\partial x}\frac{\partial w}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) + \frac{\partial^2 F}{\partial x^2}\left(\frac{\partial z}{\partial y}\frac{\partial w}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right)\right. \\ & \left. - \left(\frac{\partial^2 F}{\partial x \partial y}\right)\left(\frac{\partial z}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right] - \bar{p}w. \end{aligned} \tag{42}$$

Suppose that the boundary conditions are automatically satisfied. The system's Hamiltonian density can be then expressed as

$$L = W - T, \tag{43}$$

where

$$T := \frac{1}{2}\rho h\left(\frac{\partial w}{\partial t}\right)^2. \tag{44}$$

The corresponding Lagrangian-Euler equations are

$$\nabla^2\nabla^2 F+\frac{Eh}{2}\mathscr{B}(w,w)+Eh\mathscr{B}(z,w)=0, \tag{45}$$

$$D\nabla^2\nabla^2 w+\rho h\frac{\partial^2 w}{\partial t^2}=\bar{p}+\mathscr{B}(z,F)+\mathscr{B}(w,F), \tag{46}$$

where the bilinear form $\mathscr{B}(\cdot,\cdot)$ is defined as

$$\mathscr{B}(f,g):=\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 g}{\partial x^2}-2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial^2 g}{\partial x\partial y}\right). \tag{47}$$

Let $x_1=x$, $x_2=y$, and $x_3=t$. By using the standard abbreviated notation, one can verify that

$$\frac{\partial L}{\partial F_{,i}}=0, \quad i=1,2,3. \tag{48}$$

Introduce the two-dimensional permutation symbol as

$$e_{\alpha\beta}:=e_{3\alpha\beta} \quad \text{or} \quad e_{\alpha\beta}:=e_{\alpha\beta 3} \quad \alpha,\beta=1,2 \tag{49}$$

i.e.,

$$e_{11}=e_{22}=0, \quad e_{12}=-e_{21}=1. \tag{50}$$

It can then be shown that

$$\frac{\partial L}{\partial w_{,\alpha}}=e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}(w_{,\zeta}+z_{,\zeta}), \quad \alpha,\beta,\zeta,\eta=1,2 \tag{51}$$

and

$$\frac{\partial L}{\partial w_{,3}}=-\rho h w_{,3}. \tag{52}$$

Moreover,

$$\frac{\partial L}{\partial F_{,\alpha\beta}}=-e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}, \quad \alpha,\beta,\zeta,\eta=1,2 \tag{53}$$

$$\frac{\partial L}{\partial F_{,3i}}=0, \quad i=1,2,3 \tag{54}$$

and

$$\frac{\partial L}{\partial w_{,\alpha\beta}} = -M_{\alpha\beta}, \quad \alpha, \beta = 1, 2 \tag{55}$$

$$\frac{\partial L}{\partial w_{,3i}} = 0, \quad i = 1, 2, 3. \tag{56}$$

In order to emphasize the physical meaning, we also use the notation

$$w_{,t} := w_{,3}, \quad F_{,t} := F_{,3}, \quad Q_\alpha := M_{\alpha,\beta,\beta}, \tag{57}$$

in the sequel.

3. CONSERVATION LAWS

In this section, we shall present the main results of this study. Consider the following fundamental integral

$$J(\mathbf{q}) = \int_D L(\mathbf{y}, \mathbf{q}, \partial\mathbf{q}. \partial^2\mathbf{q})\, d\mathbf{y}, \tag{58}$$

where $y \in \mathbb{R}^2 \times \mathbb{R}_+$, $D := \Omega_p \times [0, T]$, $\Omega \in \mathbb{R}^2$. By letting

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = t; \tag{59}$$

$$q^1 = F(x_1, x_2, t), \quad q^2 = w(x_1, x_2, t), \tag{60}$$

the Lagrangain density L is assumed in the form of (43), (42), and (44). The corresponding Euler-Lagrangain equations,

$$E_\lambda^{(2)} := \frac{\partial L}{\partial q^\lambda} - \frac{\partial}{\partial y_i}\frac{\partial L}{\partial q^\lambda_{,i}} + \frac{\partial^2}{\partial y_i \partial y_j}\frac{\partial L}{\partial q^\lambda_{,ij}} = 0 \tag{61}$$

are (45)–(46). For the superscript or subscript, the Greek letters range from 1 to 2, and the Latin letters range from 1 to 3.

Denote $\mathscr{V} \subset C_2^4(D)$, $\mathscr{T} \subset C_2^3(D)$, $\mathscr{L} \subset C_2^2(D)$. We assert the following theorem.

Theorem 3.1. Let $(\mathbf{y}, \mathbf{q}, \partial\mathbf{q}, \partial^2\mathbf{q}) \in D \times \mathscr{V} \times \mathscr{T} \times \mathscr{L}$ and $\mathbf{q}$ being the solution of the Euler-Lagrangain eqns (45)–(46). The following conservation laws, (i)–(iv) hold true.

(i)
$$\frac{\partial}{\partial t}(W+T) - \frac{\partial}{\partial x_\alpha}\{e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}(w_{,\zeta}+z_{,\zeta})w_{,t} - e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}F_{,\beta t} - M_{\alpha\beta}w_{,\beta t} + e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}F_{,t} + Q_\alpha w_{,t}\} = 0; \tag{62}$$

(iia)
$$\frac{\partial}{\partial t}(\rho h w_{,t}) - \frac{\partial}{\partial x_\alpha}\{e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}(w_{,\zeta}+z_{,\zeta}) + Q_\alpha\} = 0, \quad \bar{p} = 0; \tag{63}$$

(iib)
$$\frac{\partial}{\partial x_\alpha}\{e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta,\beta}\} = 0; \tag{64}$$

(iii) $$\frac{\partial}{\partial t}(\rho h w_{,t} w_{,\gamma}\delta_{\gamma s}) + \frac{\partial}{\partial x_\alpha}\{L\delta_{\alpha s} - e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}(w_{,\zeta}+z_{,\zeta})w_{,\gamma}\delta_{\gamma s}$$
$$+ e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}F_{,\beta\eta}\delta_{\gamma s} + M_{\alpha\beta}w_{,\beta\gamma}\delta_{\gamma s} - e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta,\beta}F_{,\gamma}\delta_{\gamma s} - Q_\alpha w_{,\gamma}\delta_{\gamma s}\} = 0; \quad (65)$$

(iv) $$\frac{\partial}{\partial t}\{2(W+T)t + \rho h w_{,t} w_{,\gamma} x_{,\gamma}\} + \frac{\partial}{\partial x_\alpha}\{Lx_\alpha$$
$$- e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}(w_{,\zeta}+z_{,\zeta})(w_{,\gamma}x_\gamma + 2w_{,t}t) + e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}(F_{,\gamma\beta}x_\gamma + F_{,\beta} + 2F_{,t\beta}t)$$
$$+ M_{\alpha\beta}(w_{\gamma\beta}x_{,\gamma} + w_{,\beta} + 2w_{,t\beta}t) - e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta,\beta}(F_{,\gamma}x_\gamma + 2F_{,t}t)$$
$$- Q_\alpha(w_{,\gamma}x_\gamma + 2w_{,t}t)\} = 0, \quad \bar{p} = 0. \quad (66)$$

If the shell is isotropic, the following conservation law holds,

(v) $$\frac{\partial}{\partial t}\{\rho h w_{,t} w_{,\gamma} e_{\gamma\xi}x_\xi\} + \frac{\partial}{\partial x_\alpha}\{Le_{\alpha\beta}x_\beta - e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}(w_{,\zeta}+z_{,\zeta})w_{,\gamma}e_{\gamma\xi}x_\xi$$
$$+ e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}(F_{,\gamma\beta}e_{\gamma\xi}x_\xi + F_{,\gamma}e_{\gamma\beta}) + M_{\alpha\beta}(w_{,\gamma\beta}e_{\gamma\xi}x_\xi + F_{,\gamma}e_{\gamma\beta}) - e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta,\beta}F_{,\gamma}e_{\gamma\xi}x_\xi - Q_\alpha w_\gamma e_{\gamma\xi}x_\xi\} = 0. \quad (67)$$

Let $\Omega_0 \subset \Omega_p$. The above conservation laws can be put into the integral forms, i.e., $\forall t \in [0, T]$,

(i) $$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0}(W+T)\,\mathrm{d}\Omega_0 - \oint_{\partial\Omega_0}\{e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}(w_{,\zeta}+z_{,\zeta})w_{,t} - e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}F_{,\beta t}$$
$$- M_{\alpha\beta}w_{,\beta t} + e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}F_{,t} + Q_\alpha w_{,t}\}n_\alpha\,\mathrm{d}\Gamma = 0; \quad (68)$$

(iia) $$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0}(\rho h w_{,t})\,\mathrm{d}\Omega_0 - \oint_{\partial\Omega_0}\{e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}(w_{,\zeta}+z_{,\zeta}) + Q_\alpha\}n_\alpha\,\mathrm{d}\Gamma = 0, \quad \bar{p} = 0; \quad (69)$$

(iib) $$\oint_{\partial\Omega_0}\{e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta,\beta}\}n_\alpha\,\mathrm{d}\Gamma = 0; \quad (70)$$

(iii) $$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0}(\rho h w_{,t} w_{,\gamma}\delta_{\gamma s})\,\mathrm{d}\Omega_0 + \oint_{\partial\Omega_0}\{L\delta_{\alpha s} - e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}(w_{,\zeta}+z_{,\zeta})w_{,\gamma}\delta_{\gamma s}$$
$$+ e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}F_{,\beta\eta}\delta_{\gamma s} + M_{\alpha\beta}w_{,\beta\gamma}\delta_{\gamma s} - e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta,\beta}F_{,\gamma}\delta_{\gamma s} - Q_\alpha w_{,\gamma}\delta_{\gamma s}\}n_\alpha\,\mathrm{d}\Gamma = 0; \quad (71)$$

(iv) $$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0}\{2)(W+T)t + \rho h w_{,t} w_{,\gamma} x_\gamma\}\,\mathrm{d}\Omega_0$$
$$+ \oint_{\partial\Omega_0}\{Lx_\alpha - e_{\alpha\beta}e_{\zeta\eta}F_{\beta\eta}(w_{,\zeta}+z_{,\zeta})(w_{,\gamma}x_\gamma + 2w_{,t}t) + e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}(F_{,\gamma\beta}x_\gamma + F_{,\beta} + 2F_{,t\beta}t)$$
$$+ M_{\alpha\beta}(w_{,\gamma\beta}x_{,\gamma} + w_{,\beta} + 2w_{,t\beta}t) - e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta,\beta}(F_{,\gamma}x_\gamma + 2F_{,t}t)$$
$$+ Q_\alpha(w_{,\gamma}x_\gamma + 2w_{,t}t)\}n_\alpha\,\mathrm{d}\Gamma = 0, \quad \bar{p} = 0; \quad (72)$$

(v) $$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0}\{\rho h w_{,t} w_{,\gamma} e_{\gamma\xi}x_\xi\}\,\mathrm{d}\Omega_0 + \oint_{\partial\Omega_0}\{Le_{\alpha\beta}x_\beta - e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}(w_{,\zeta}+z_{,\zeta})w_{,\gamma}e_{\gamma\xi}x_\xi$$
$$+ e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}(F_{,\gamma\beta}e_{\gamma\xi}x_\xi + F_{,\gamma}e_{\gamma\beta}) + M_{\alpha\beta}(w_{,\gamma\beta}e_{\gamma\xi}x_\xi + F_{,\gamma}e_{\gamma\beta})$$
$$- e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\eta,\beta}F_{,\gamma}e_{\gamma\xi}x_\xi - Q_\alpha w_\gamma e_{\gamma\xi}x_\xi\}n_\alpha\,\mathrm{d}\Gamma = 0. \quad (73)$$

Before proceeding to the proof, a few remarks are in order. Conservation laws (i) and (ii) are the consequences of the conservation of energy and conservation of linear momentum respectively. For the Marguerre-von Kármán shallow shell, the in-plane inertia is neglected (Chia, 1977), whence there is no time derivative terms involved in conservation law (iib). By the assumption that the projection of the deformed middle surface of the nonlinear shallow shell onto the x–y plane is always the same, the primary unknown variables, the Airy stress function F and the shell deflection w, are not perceptible to the in-plane rigid-body rotation. Thus, no conservation laws correspond to the conservation of the angular momentum.

The conservation laws, (iii)–(v), are the counterparts to the classic conservation laws of **J**, M, and **L** integrals. In the Marguerre-von Kármán shallow shell theory, these three types of invariant integrals can be expressed as follows

$$J_s := \oint_\Gamma \{L\delta_{\alpha s} - e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}(w_{\zeta}+z_{,\zeta})w_{,\gamma}\delta_{\gamma s} + e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}F_{,\beta\eta}\delta_{\gamma s} + M_{\alpha\beta}w_{,\beta\gamma}\delta_{\gamma s} - e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta,\beta}F_{,\gamma}\delta_{\gamma s} - Q_\alpha w_{,\gamma}\delta_{\gamma s}\}n_\alpha \,\mathrm{d}\Gamma; \quad (74)$$

$$L = \oint_\Gamma \{Le_{\alpha\beta}x_\beta - e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}(w_{,\zeta}+z_{,\zeta})w_{,\gamma}e_{\gamma\xi}x_\xi + e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}(F_{,\gamma\beta}e_{\gamma\xi}x_\xi + F_{,\gamma}e_{\gamma\beta}) + M_{\alpha\beta}(w_{,\gamma\beta}e_{\gamma\xi}x_\xi + F_{,\gamma}e_{\gamma\beta}) - e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta,\beta}F_{,\gamma}e_{\gamma\xi}x_\xi - Q_\alpha w_\gamma e_{\gamma\xi}x_\xi\}n_\alpha \,\mathrm{d}\Gamma; \quad (75)$$

$$M = \oint_\Gamma \{Lx_\alpha - e_{\alpha\beta}e_{\zeta\eta}F_{\beta\eta}(w_{,\zeta})(w_{,\gamma}x_\gamma + 2w_{,t}t) + e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta}(F_{,\gamma\beta}x_\gamma + F_{,\beta} + 2F_{,t\beta}t) + M_{\alpha\beta}(w_{,\gamma\beta}x_{,\gamma} + w_{,\beta} + 2w_{,t\beta}t) - e_{\alpha\zeta}e_{\beta\eta}\varepsilon^{(0)}_{\zeta\eta,\beta}(F_{,\gamma}x_\gamma + 2F_{,t}t) + Q_\alpha(w_{,\gamma}x_\gamma + 2w_{,t}t)\}n_\alpha \,\mathrm{d}\Gamma. \quad (76)$$

It is arguable, however, whether or not the M integral (76) is still invariant in the static case.

Proof: (i) The invariant transformation in this case is the time translation. Let

$$\bar{y}_\alpha = y_\alpha = x_\alpha, \quad \bar{y}_3 = t+\varepsilon, \quad \text{and} \quad \bar{q}^\lambda = q^\lambda. \quad (77)$$

It is obvious that the integral (58) is invariant, i.e., $J(\bar{\mathbf{q}}, \partial\bar{\mathbf{q}}, \partial^2\bar{\mathbf{q}}) = J(\mathbf{q}, \partial\mathbf{q}. \partial^2\mathbf{q})$.

Substitute

$$\tau^\alpha_s = \left.\frac{\partial\phi_\alpha}{\partial\varepsilon}\right|_{\varepsilon=0} = 0, \quad \tau^3_s = \left.\frac{\partial\phi_3}{\partial\varepsilon}\right|_{\varepsilon=0} = 1, \quad \xi^\lambda_s = 0, \quad (78)$$

and

$$C^1_s = -F_{,t}, \quad C^2_s = -w_{,t}, \quad (79)$$

into (13). Then the conservation law (i) follows immediately.

(ii) In this case, the transformation is rigid-body translation, i.e.,

$$\bar{y}_i = y_i, \quad \bar{p} = p\delta_{1s}, \quad \text{and} \quad \bar{q}^\lambda = q^\lambda + \varepsilon\delta_{\lambda s}. \quad (80)$$

One can show that integral (58) is invariant under the above transformation. However, there is a constraint imposed on the normal distribution load $\bar{p}$. Substitution of

$$\tau_s^i = 0, \quad \xi_s^\lambda = \delta_{\lambda s}, \quad C_s^\lambda = \delta_{\lambda s} \tag{81}$$

into (13) furnishes the conservation law (ii).

(iii) Consider the following spatial coordinate transformation,

$$\bar{y}_\alpha = y_\alpha + \varepsilon\delta_{\alpha s}, \quad \bar{y}_3 = y_3, \quad \bar{q}^\lambda = q^\lambda. \tag{82}$$

It is quite obvious that $J(\bar{\mathbf{q}}, \partial\bar{\mathbf{q}}, \partial^2\bar{\mathbf{q}}) = J(\mathbf{q}, \partial\mathbf{q}, \partial^2\mathbf{q})$. Subsequently, it follows that

$$\tau_s^\alpha = \delta_{\alpha s}, \quad \tau_s^3 = 0, \quad \xi_s^\lambda = 0, \quad C_s^\lambda = -q_{,\gamma}^\lambda \delta_{\gamma s}. \tag{83}$$

Making these substitutions in (13), one may obtain the conservation law (iii).

(iv) Let

$$\bar{y}_\alpha = (1+\varepsilon)y_\alpha, \quad \bar{y}_3 = (1+2\varepsilon)t, \quad \bar{q}^\lambda = q^\lambda, \quad \text{and} \quad \bar{p} = 0. \tag{84}$$

Under the above dilatation transformation, the fundamental integral (58) is no more invariant, but it is still infinitesimally invariant, i.e.,

$$J(\bar{\mathbf{q}}, \partial\bar{\mathbf{q}}, \partial^2\bar{\mathbf{q}}) = J(\mathbf{q}, \partial\mathbf{q}. \partial^2\mathbf{q}) + o(\varepsilon). \tag{85}$$

One can find that

$$\tau_s^\alpha = x_\alpha, \quad \tau_s^3 = 2t, \quad \xi_s^\lambda = 0, \tag{86}$$

furthermore

$$C_s^1 = -F_{,\gamma}x_\gamma - 2F_{,t}t, \quad C_s^2 = -w_{,\gamma}x_\gamma - 2w_{,t}t. \tag{87}$$

Substitution the above expressions into eqn (13) yields the conservation law (iv). Once again, the conservation law holds, if only the normal external load $\bar{p} = 0$; otherwise, an additional surface integral will be involved.

(v) For isotropic, elastic shell, the fundamental integral (58) is invariant under in-plane coordinate rotation, i.e.,

$$\bar{x}_\alpha = Q_{\alpha\beta}(\varepsilon)x_\beta, \tag{88}$$

$$\bar{q}^\lambda = q^\lambda, \tag{89}$$

where $Q_{\alpha\beta}(\varepsilon)$ is the element of orthogonal transformation.

For the sake of simplicity, if $\varepsilon \ll 1$, the in-plane rotation transformation can be simplified as follows

$$\bar{y}_\alpha = y_\alpha + e_{\alpha\beta}y_\beta\varepsilon, \quad \bar{y}_3 = y_3, \quad \bar{q}^\lambda = q^\lambda. \tag{90}$$

After some elementary algebra manipulation, one may be able to show that

$$\left(\frac{\partial^2}{\partial \bar{x}_1^2}+\frac{\partial^2}{\partial \bar{x}_2^2}\right)\bar{q}^\lambda = \left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)q^\lambda + o(\varepsilon) \tag{91}$$

$$\left(\frac{\partial^2 \bar{q}^\lambda}{\partial \bar{x}_1 \partial \bar{x}_2}\right)^2 - \frac{\partial^2 \bar{q}^\lambda}{\partial \bar{x}_1^2}\frac{\partial^2 \bar{q}^\lambda}{\partial \bar{x}_2^2} = \left(\frac{\partial^2 q^\lambda}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 q^\lambda}{\partial x_1^2}\frac{\partial^2 q^\lambda}{\partial x_2^2} + o(\varepsilon), \tag{92}$$

and

$$\left[\frac{\partial^2 \bar{q}^1}{\partial \bar{x}_2^2}\left(\frac{\partial \bar{q}^2}{\partial \bar{x}_1}\right)^2 + \frac{\partial^2 \bar{q}^1}{\partial \bar{x}_1^2}\left(\frac{\partial \bar{q}^2}{\partial \bar{x}_2}\right)^2 - 2\frac{\partial^2 \bar{q}^1}{\partial \bar{x}_1 \partial \bar{x}_2}\frac{\partial \bar{q}^2}{\partial \bar{x}_1}\frac{\partial \bar{q}^2}{\partial \bar{x}_2}\right]$$
$$= \left[\frac{\partial^2 q^1}{\partial x_2^2}\left(\frac{\partial q^2}{\partial x_1}\right)^2 + \frac{\partial^2 q^1}{\partial x_1^2}\left(\frac{\partial q^2}{\partial x_2}\right)^2 - 2\frac{\partial^2 q^1}{\partial x_1 \partial x_2}\frac{\partial q^2}{\partial x_1}\frac{\partial q^2}{\partial x_2}\right] + o(\varepsilon), \tag{93}$$

whence,

$$J(\bar{\mathbf{q}}, \partial\bar{\mathbf{q}}, \partial^2\bar{\mathbf{q}}) = J(\mathbf{q}, \partial\mathbf{q}, \partial^2\mathbf{q}) + o(\varepsilon). \tag{94}$$

It can be readily shown that

$$\tau_s^\alpha = e_{\alpha\beta}x_\beta \tag{95}$$

$$C_s^\lambda = -q_{,\xi}^\lambda e_{\xi\gamma}x_\gamma \frac{\partial C_s^\lambda}{\partial x_\beta} = -q_{,\xi\beta}^\lambda e_{\xi\gamma}x_\gamma - q_{,\xi}^\lambda e_{\xi\beta}. \tag{96}$$

Substitution (95)–(96) into (13) yields the conservation law (v). ■

Remark 3.1. The only difference between the Marguerre shallow shells and the von Kármán plates is the presence of the initial deflection z. In the above proof, we have implicitly assumed that the initial deflection z to be an infinitesimal algebraic invariant function (see Olver, 1986), i.e.

$$z(\bar{\mathbf{x}}) = z(\mathbf{x}) + o(\varepsilon).$$

This is a very severe restriction on the conservation laws in categories (iii) and (iv). Nevertheless, there is still a large class of functions that fit the requirement. For example, for the conservation law (iii),

$$\bar{x}_1 = x_1 + \varepsilon, \quad \bar{x}_2 = x_2;$$
$$z = x_1^2 + x_2^2 - 2x_1 - 2x_2 + c.$$

Then

$$\begin{aligned} z(\bar{\mathbf{x}}) &= (x_1+\varepsilon)^2 + x_2^2 - 2x_1 - 2x_2 + c \\ &= x_1^2 + x_2^2 - 2x_1 - 2x_2 + c + \varepsilon^2 \\ &= z + o(\varepsilon). \end{aligned}$$

For case (iv), however, it seems that the only admissible initial deflection is constant, or fractional functions.

The fundamental integral for the Marguerre-von Kármán shallow shell is generally not invariant under dilatation transformation. This is even true in the linear plate theory (see the discussion in Sosa *et al.*, 1988). However, if we loosen our definition on "invariance", we might still be able to find some semi-invariant integrals. As pointed out by Sosa *et al.* (1988), in general, this type of integrals can be expressed in the form,

$$I = \oint_{\partial\Omega_p} L_{line}\,\mathrm{d}\Gamma + \int_{\Omega_p} L_{domain}\,\mathrm{d}\Omega. \tag{97}$$

As a matter of fact, by using the second-order Noether theorem (Section 2.2), one can construct such invariant integrals in a systematic manner. In what follows, a specific example is demonstrated for how to construct such invariant integrals.

To illustrate our point, we assert the following statement:

Theorem 3.2. Assume that the initial deflection satisfying $z(\bar{\mathbf{x}}) = z(\mathbf{x}) + o(\varepsilon)$ under the admissible infinitesimal transformation. For the Marguerre-von Kármán shallow shell, if the primary variable pair (F, w) satisfies the governing eqns (45)–(46), the following integral is invariant,

$$\begin{aligned}\mathscr{M} := \oint_{\partial\Omega_0} & (L_1 x_\alpha + e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}z_{,\zeta}(w - w_{,\gamma}x_\gamma) + e_{\alpha\beta}e_{\zeta\eta}\varepsilon_{\zeta\eta 0}F_{,\beta\gamma}x_\gamma + M_{\alpha\beta}w_{,\beta\gamma}x_\gamma \\ & + e_{\alpha\beta}e_{\zeta\eta}\varepsilon_{\zeta\eta 0,\beta}(F - F_{,\gamma}x_\gamma) + Q_\alpha(w - w_{,\gamma}x_\gamma))n_\alpha\,\mathrm{d}\Gamma \\ & + \int_{\Omega_0}\left(\frac{1}{2}\mathscr{B}(w,w)(F - F_{,\gamma}x_\gamma) + (\mathscr{B}(w,F) + \bar{p} - \rho h w_{,tt})(w - w_{,\gamma}x_\gamma)\right)\mathrm{d}\Omega_0 = 0,\end{aligned} \tag{98}$$

where

$$\begin{aligned}L_1(\partial\mathbf{q}, \partial^2\mathbf{q}) := & -\frac{1}{2Eh}\left[\left(\frac{\partial^2 q^1}{\partial x_1^2} + \frac{\partial^2 q^1}{\partial x_2^2}\right)^2 + 2(1+\nu)\left(\frac{\partial^2 q^1}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 q^1}{\partial x_1^2}\frac{\partial^2 q^1}{\partial x_2^2}\right] \\ & + \frac{D}{2}\left[\left(\frac{\partial^2 q^2}{\partial x_1^2} + \frac{\partial^2 q^2}{\partial x_2^2}\right)^2 + 2(1-\nu)\left(\frac{\partial^2 q^2}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 q^2}{\partial x_1^2}\frac{\partial^2 q^2}{\partial x_2^2}\right] \\ & + \left[\frac{\partial^2 q^1}{\partial x_2^2}\frac{\partial z}{\partial x_1}\frac{\partial q^2}{\partial x_1} + \frac{\partial^2 q^1}{\partial x_1^2}\frac{\partial z}{\partial x_2}\frac{\partial q^2}{\partial x_2} - \frac{\partial^2 q^1}{\partial x_1 \partial x_2}\left(\frac{\partial z}{\partial x_2}\frac{\partial q^2}{\partial x_1} + \frac{\partial z}{\partial x_1}\frac{\partial q^2}{\partial x_2}\right)\right].\end{aligned} \tag{99}$$

Note that $\varepsilon_{\zeta\eta 0} \neq \varepsilon^{(0)}_{\zeta\eta}$. Here,

$$\varepsilon_{\zeta\eta 0} := \tfrac{1}{2}(u^{(0)}_{\zeta,\eta} + u^{(0)}_{\eta,\zeta} + z_{,\zeta}w_{,\eta} + z_{,\eta}w_{,\zeta}). \tag{100}$$

Before proceeding to the proof, one may verify the following equalities,

$$\frac{\partial L_1}{\partial F_{,i}} = 0, \tag{101}$$

$$\frac{\partial L_1}{\partial w_{,\alpha}} = e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}z_{,\zeta}, \tag{102}$$

$$\frac{\partial L_1}{\partial F_{,\alpha\beta}} = -e_{\alpha\beta}e_{\zeta\eta}\varepsilon_{\zeta\eta 0}, \tag{103}$$

$$\frac{\partial L_1}{\partial w_{,\alpha\beta}} = -M_{\alpha\beta}. \tag{104}$$

Proof: Split the Lagrangian density (43) into two parts,

$$L(\mathbf{q}, \partial\mathbf{q}, \partial^2\mathbf{q}) = L_1(\partial^2\mathbf{q}) + L_2(\mathbf{q}, \partial\mathbf{q}, \partial^2\mathbf{q}), \tag{105}$$

where L_1 is defined in (99) and L_2 is defined as follows

$$L_2 := \frac{1}{2}\left[\frac{\partial^2 q^1}{\partial x_2^2}\left(\frac{\partial q^2}{\partial x_1}\right)^2 + \frac{\partial^2 q^1}{\partial x_1^2}\left(\frac{\partial q^2}{\partial x_2}\right)^2 - 2\frac{\partial^2 q^1}{\partial x_1 \partial x_2}\frac{\partial q^2}{\partial x_1}\frac{\partial q^2}{\partial x_2}\right] - \bar{p}q^2 - \frac{1}{2}\rho h\left(\frac{\partial q^2}{\partial t}\right)^2. \tag{106}$$

Under the infinitesimal dilatation transformation,

$$\phi = (1+\varepsilon)\mathbf{x}, \tag{107}$$

$$\psi = (1+\varepsilon)\mathbf{q}, \tag{108}$$

and the assumption,

$$z(\bar{\mathbf{q}}) = z(\mathbf{q}) + o(\varepsilon), \tag{109}$$

it is not difficult to show that

$$J_1(\mathbf{q}) = \int_{\Omega_p} L_1(\partial\mathbf{q}, \partial^2\mathbf{q})\, \mathrm{d}\Omega \tag{110}$$

is infinitesimally invariant, i.e.,

$$J_1(\bar{\mathbf{q}}) = J_1(\mathbf{q}) + o(\varepsilon). \tag{111}$$

Based on the definitions, one may find that

$$\tau_s^\alpha = \left.\frac{\partial \phi^\alpha}{\partial \varepsilon}\right|_{\varepsilon=0} = x_\alpha \tag{112}$$

$$\xi_s^\lambda = \left.\frac{\partial \psi^\lambda}{\partial \varepsilon}\right|_{\varepsilon=0} = q^\lambda \tag{113}$$

$$C_s^\lambda = \xi_s^\lambda - q_{,\gamma}^\lambda \tau_s^\gamma = q^\lambda - \frac{\partial q^\lambda}{\partial x_\gamma} x_\gamma. \tag{114}$$

Thus, utilizing the Noether theorem (Section 2.2), one has

$$\frac{\partial}{\partial x_\alpha}\left[L_1 x_\alpha + \frac{\partial L_1}{\partial q_{,\alpha}^\lambda} C_s^\lambda + \frac{\partial L_1}{\partial q_{,\alpha\beta}^\lambda}\frac{\mathrm{d}C_s^\lambda}{\mathrm{d}x_\beta} - \frac{\partial}{\partial x_\beta}\frac{\partial L}{\partial q_{,\alpha\beta}^\lambda} C_s^\lambda\right] = -E_{1\lambda}^{(2)} C_s^\lambda. \tag{115}$$

However,

$$E_{1\lambda}^{(2)} := \frac{\partial L_1}{\partial q^\lambda} - \frac{\partial}{\partial x_\alpha}\frac{\partial L_1}{\partial q^\lambda_{,\alpha}} + \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\frac{\partial L_1}{\partial q^\lambda_{,\alpha\beta}} \neq 0, \tag{116}$$

instead

$$E_{11}^{(2)} = -\frac{1}{Eh}\nabla^2\nabla^2 q^1 - \mathscr{B}(z, q^2) = \frac{1}{2}\mathscr{B}(q^2, q^2), \tag{117}$$

$$E_{12}^{(2)} = D\nabla^2\nabla^2 q^2 - \mathscr{B}(z, q^1) = \bar{p} - \rho h\frac{\partial^2 q^2}{\partial t^2} + \mathscr{B}(q^1, q^2). \tag{118}$$

Recall $q^1 := F$, $q^2 := w$. One can show the following semi-conservative equation,

$$\begin{aligned}\frac{\partial}{\partial x_\alpha}\{&L_1 x_\alpha + e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}z_{,\zeta}(w - w_{,\gamma}x_\gamma) + e_{\alpha\beta}e_{\zeta\eta}\varepsilon_{\zeta\eta 0}F_{,\beta\gamma}x_\gamma + M_{\alpha\beta}w_{,\beta\gamma}x_\gamma \\ &+ e_{\alpha\beta}e_{\zeta\eta}\varepsilon_{\zeta\eta 0,\beta}(F - F_{,\gamma}x_\gamma) + Q_\alpha(w - w_{,\gamma}x_\gamma)\} = \\ -\Big(&\frac{1}{2}B(w, w)(F - F_{,\gamma}x_\gamma) + (B(w, F) - \bar{p} - \rho h w_{,tt})(w - w_{,\gamma}x_\gamma)\Big),\end{aligned} \tag{119}$$

which leads to (98).

If one neglects the nonlinear deformation, dynamic effects, and the external load, a truly path-independent integral holds for the linear shallow shells, i.e.,

$$\begin{aligned}\mathscr{M} = \oint_{\partial\Omega_0}\{&L_1 x_\alpha + e_{\alpha\beta}e_{\zeta\eta}F_{,\beta\eta}z_{,\zeta}(w - w_{,\gamma}x_\gamma) + e_{\alpha\beta}e_{\zeta\eta}\varepsilon_{\zeta\eta 0}F_{,\beta\gamma}x_\gamma + M_{\alpha\beta}w_{,\beta\gamma}x_\gamma \\ &+ e_{\alpha\beta}e_{\zeta\eta}\varepsilon_{\zeta\eta 0,\beta}(F - F_{,\gamma}x_\gamma) + Q_\alpha(w - w_{,\gamma}x_\gamma)\}n_\alpha\,\mathrm{d}\Gamma = 0.\end{aligned} \tag{120}$$

■

4. CLOSURE

In this study, several conservation laws for the Marguerre-von Kármán shallow shell theory have been derived. Since the Lagrangian density concerned here belongs to a mixed variational functional, which is involved with both deflection and stress function; thus, the corresponding conservation laws may have inherent difference from those derived from the energy based variational principles. Based on this fact, one might speculate that the J-integral obtained here may be related to a mixed-mode energy release rate. With the help of these new path-independent integrals, the asymptotic field around a stress concentrated area in the shell can be evaluated and estimated with convenience.

This work was done 3 years ago. Recently, we found a paper by Djondjorov *et al.* (1996), which also deals with the conservation laws for von Kármán plate. In that paper, by using Lie group technique, a complete list of conservation laws has been given. Nevertheless, the results presented here not only offer a list of all the important conservation laws for Marguerre-von Kármán shallow shell, but also provide much more explicit expressions for these conservation laws, such that they can be easily applied to practical engineering problems.

By employing the second order Noether theorem on covariant field, e.g., Blakeslee and Logan (1977), the same procedure can be used to derive conservation laws for general thin shell theory, which is subjected to further study.

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